

ON SYMPLECTIC DYNAMICS

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ABSTRACT. This paper continues to carry out a foundational study of Banyaga's topologies of a closed symplectic manifold [3]. Our intension in writing this paper is to provide several symplectic analogues of some results found in the study of Hamiltonian dynamics. Especially, without appealing to the positivity of the symplectic displacement energy, we point out the impact of the L^∞ version of Banyaga's Hofer-like metric in the investigation of the symplectic nature of the C^0 -limit of a sequence of symplectic maps. This result is the symplectic analogue of a result that was proved by Hofer-Zehnder [8] (for compactly supported Hamiltonian diffeomorphisms on \mathbb{R}^{2n}), and then reformulated by Oh-Müller [10] for Hamiltonian diffeomorphisms in general. Furthermore, we extend to symplectic isotopies the regularization procedure for Hamiltonian paths introduced by Polterovich [11], and then we use it to prove the equality between the two versions of Banyaga's Hofer-like norms defined on the identity component in the group of symplectomorphisms. This result was announced in [2]. It shows the uniqueness of Banyaga's Hofer-like geometry, and then yields the symplectic analogue of a result that was proved by Polterovich [11]. Finally, we elaborate the symplectic analogues of some approximation results found in Oh-Müller [10], and make some remarks on flux theory.

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1. INTRODUCTION

The Hofer geometry started with the remarkable paper of Hofer [7] that introduced the Hofer topologies on the space of Hamiltonian diffeomorphisms of a symplectic manifold. These topologies motivated various studies in the field of Hamiltonian dynamics. In particular, Hofer-Zehnder [8] derived almost all the basic formulae and some perspectives for the subsequent development of Hamiltonian dynamics. A thorough discussion of Hofer topologies can be found in [8, 9, 10, 11].

Recently, it was shown by Banyaga [3] that the Hofer topologies admit a natural generalization to the set of all time-one maps of symplectic isotopies of a symplectic manifold. In particular, if a symplectic manifold is such that the identity component in its group of symplectic diffeomorphisms is reduced to the group of all Hamiltonian diffeomorphisms, then the corresponding Banyaga's topologies [3] reduce to Hofer's topologies.

These facts attest that it is judicious to investigate whether the analogues of some results found in the study of Hamiltonian dynamics can be elaborated in the context of Banyaga's Hofer-like geometry or not.

This motivated the results of the present paper: In Section 2, we recall some fundamental facts concerning symplectic mappings and isotopies. Section 2.8 introduces Hopf-Rinow theorem from Riemannian geometry and shows its implication in the study of Hofer's norms with respect to a certain class of functions. In Section 2.9, using Hodge's theory, we show that Polterovich's regularization process for Hamiltonian isotopies admits a natural generalization to symplectic isotopies. Section 3 deals with the main results of the present

paper. Here, studying Banyaga's topologies, we use the results of Sections 2.8 and 2.9 to prove that Banyaga's Hofer like-geometry is invariant under the choice of Banyaga's Hofer-like norm. We show an impact of the L^∞ version of the Hofer-like metric in the investigation of the symplectic nature of a homeomorphism which is the C^0 -limit of a sequence of symplectic diffeomorphisms. This follows by combining Hodge's decomposition theorem of symplectic isotopies together with the standard continuity theorem of ODE for Lipschitz vector fields. Furthermore, we prove that if a loop is homotopic (relatively to a fixed base point) to a closed Hamiltonian orbit, then the symplectic area swept by the latter under the symplectic flow generated by any non exact closed 1-form is trivial. This can be viewed (in a certain sense) as the dual form of a following well-known result from flux geometry. Section 4 contains some approximation lemmas that generalize some results found in the study of Hamiltonian dynamics.

2. PRELIMINARIES

Let M be a smooth closed manifold of dimension $2n$. A differential 2-form ω on M is called a symplectic form if ω is closed and nondegenerate. In particular, any symplectic manifold is oriented. From now on, we shall always assume that M admits a symplectic form ω . A diffeomorphism $\phi : M \rightarrow M$ is called symplectic if it preserves the symplectic form ω .

2.1. Symplectic vector fields. The symplectic structure ω on M , being nondegenerate, induces an isomorphism between vector fields Z and 1-forms on M given by $Z \mapsto \iota(Z)\omega$. A vector field Z on M is symplectic if $\iota(Z)\omega$ is closed. In particular, a symplectic vector field Z on M is said to be a Hamiltonian vector field if $\iota(Z)\omega$ is exact. It follows from the definition of symplectic vector fields that, if the first de Rham cohomology group of the manifold M is trivial (i.e. $H^1(M, \mathbb{R}) = 0$), then all the symplectic vector fields induced by a symplectic form ω on M are Hamiltonian. If we equip M with a Riemannian metric g (any differentiable manifold M can be equipped with a Riemannian metric), then any harmonic 1-form α on M determines a symplectic vector field Z such that $\iota(Z)\omega = \alpha$ (so-called harmonic vector field, see [3]). In view of Hodge's theory, a sufficient condition that guarantees the existence of nontrivial harmonic vector fields on a symplectic manifold (M, ω) is that $H^1(M, \mathbb{R}) \neq 0$. Note that $H^1(M, \mathbb{R})$ is a topological invariant, i.e. it does not depend on the differentiable structure on M and depends only on the underlying topological structure of M [13].

2.2. Symplectic isotopies. An isotopy $\{\phi_t\}$ of the symplectic manifold (M, ω) is said to be symplectic if for each t , the vector field $Z_t = \frac{d\phi_t}{dt} \circ \phi_t^{-1}$ is symplectic. In particular a symplectic isotopy $\{\psi_t\}$ is called Hamiltonian if for each t , $Z_t = \frac{d\psi_t}{dt} \circ \psi_t^{-1}$ is Hamiltonian, i.e. there exists a smooth function $F : [0, 1] \times M \rightarrow \mathbb{R}$ called Hamiltonian such that $\iota(Z_t)\omega = dF_t$. As we can see, any Hamiltonian isotopy determines a Hamiltonian $F : [0, 1] \times M \rightarrow \mathbb{R}$ up to an additive constant. Throughout the paper we assume that all Hamiltonians are normalized in the following way: given a Hamiltonian $F : [0, 1] \times M \rightarrow \mathbb{R}$ we require that $\int_M F_t \omega^n = 0$. We denote by $\mathcal{N}([0, 1] \times M, \mathbb{R})$ the space of all smooth normalized Hamiltonians and by $Ham(M, \omega)$ the set of all time-one maps of Hamiltonian isotopies. If we equip M with a Riemannian metric g , then a symplectic isotopy $\{\theta_t\}$ is said to be harmonic if for each t , $Z_t = \frac{d\theta_t}{dt} \circ \theta_t^{-1}$ is harmonic. We

denote by $Iso(M, \omega)$ the group of all symplectic isotopies of (M, ω) and by $Symp_0(M, \omega)$ the set of all time-one maps of symplectic isotopies.

2.3. Harmonics 1–forms. From now on, we assume that M is equipped with a Riemannian metric g , and denote by $\mathcal{H}^1(M, g)$ the space of harmonic 1–forms on M with respect to the Riemannian metric g . In view of the Hodge theory, $\mathcal{H}^1(M, g)$ is a finite dimensional vector space over \mathbb{R} which is isomorphic to $H^1(M, \mathbb{R})$ (see [13]). The dimension of $\mathcal{H}^1(M, g)$ is the first Betti number of the manifold M , denoted by b_1 . Taking $(h_i)_{1 \leq i \leq b_1}$ as a basis of the vector space $\mathcal{H}^1(M, g)$, we equip $\mathcal{H}^1(M, g)$ with the Euclidean norm $|\cdot|$ defined as follows : for all H in $\mathcal{H}^1(M, g)$ with $H = \sum_{i=1}^{b_1} \lambda_i h_i$ we have

$$|H| := \sum_{i=1}^{b_1} |\lambda_i|.$$

It is convenient to compare the above Euclidean norm with the well-known uniform sup norm of differential 1–forms. For this purpose, let's recall the definition of the uniform sup norm of a differential 1–form α on M . For all $x \in M$, we know that α induces a linear map $\alpha_x : T_x M \rightarrow \mathbb{R}$ whose norm is given by

$$\|\alpha_x\| = \sup\{|\alpha_x(X)| : X \in T_x M, \|X\|_g = 1\},$$

where $\|\cdot\|_g$ is the norm induced on each tangent space $T_x M$ by the Riemannian metric g . Therefore, the uniform sup norm of α , say $|\cdot|_0$ is defined by $|\alpha|_0 = \sup_{x \in M} \|\alpha_x\|$. In particular, when α is a harmonic 1–form (i.e $\alpha = \sum_{i=1}^{b_1} \lambda_i h_i$), we obtain the following estimates,

$$|\alpha|_0 \leq \sum_{i=1}^{b_1} |\lambda_i| |h_i|_0 \leq E |\alpha|,$$

where $E := \max_{1 \leq i \leq b_1} |h_i|_0$. If the basis $(h_i)_{1 \leq i \leq b_1}$ is such that $E > 1$, then one can always normalize such a basis so that E equals 1. Otherwise, the identity $|\alpha|_0 \leq E |\alpha|$ reduces to $|\alpha|_0 \leq |\alpha|$. We denote by $\mathcal{PH}^1(M, g)$, the space of smooth mappings $\mathcal{H} : [0, 1] \rightarrow \mathcal{H}^1(M, g)$.

2.4. A description of symplectic isotopies [2]. In this subsection, from the group of symplectic isotopies, we shall deduce another group which will be convenient later on (see [2]).

Consider $\{\phi_t\}$ to be a symplectic isotopy, for each t , the vector field $Z_t = \frac{d\phi_t}{dt} \circ (\phi_t)^{-1}$ satisfies $d\iota(Z_t)\omega = 0$. So, it follows from Hodge's theory that $\iota(Z_t)\omega$ decomposes as the sum of an exact 1–form dU_t^Φ and a harmonic 1–form \mathcal{H}_t^Φ (see [13]). Denote by U the Hamiltonian $U^\Phi = (U_t^\Phi)$ normalized, and by \mathcal{H} the smooth family of harmonic 1–forms $\mathcal{H}^\Phi = (\mathcal{H}_t^\Phi)$. In [2], the authors denoted by $\mathfrak{T}(M, \omega, g)$ the Cartesian product $\mathcal{N}([0, 1] \times M, \mathbb{R}) \times \mathcal{PH}^1(M, g)$, and equipped it with a group structure which makes the bijection

$$(2.1) \quad Iso(M, \omega) \rightarrow \mathfrak{T}(M, \omega, g), \Phi \mapsto (U, \mathcal{H})$$

a group isomorphism. Denoting the map just constructed by \mathfrak{A} , the authors denoted any symplectic isotopy $\{\phi_t\}$ as $\phi_{(U, \mathcal{H})}$ to mean that the mapping \mathfrak{A} maps $\{\phi_t\}$ onto (U, \mathcal{H}) , and (U, \mathcal{H}) is called the “generator” of the symplectic path $\phi_{(U, \mathcal{H})}$. In particular, any symplectic isotopy of the form $\phi_{(0, \mathcal{H})}$ is considered to be a harmonic isotopy, while any symplectic isotopy of the form $\phi_{(U, 0)}$ is considered to be a Hamiltonian isotopy. The product in $\mathfrak{T}(M, \omega, g)$ is given by,

$$(2.2) \quad (U, \mathcal{H}) \rtimes (V, \mathcal{K}) = (U + V \circ \phi_{(U, \mathcal{H})}^{-1} + \tilde{\Delta}(\mathcal{K}, \phi_{(U, \mathcal{H})}^{-1}), \mathcal{H} + \mathcal{K})$$

The inverse of (U, \mathcal{H}) , denoted $\overline{(U, \mathcal{H})}$ is given by

$$(2.3) \quad \overline{(U, \mathcal{H})} = (-U \circ \phi_{(U, \mathcal{H})} - \tilde{\Delta}(\mathcal{H}, \phi_{(U, \mathcal{H})}), -\mathcal{H})$$

where for each t , $\phi_{(U, \mathcal{H})}^{-t} := (\phi_{(U, \mathcal{H})}^t)^{-1}$, and $\tilde{\Delta}_t(\mathcal{K}, \phi_{(U, \mathcal{H})}^{-1})$ is the function $\Delta_t(\mathcal{K}, \phi_{(U, \mathcal{H})}^{-1}) := \int_0^t \mathcal{K}_s(\dot{\phi}_{(U, \mathcal{H})}^{-s}) \circ \phi_{(U, \mathcal{H})}^{-s} ds$ normalized.

Here is a consequence of the Hodge decomposition theorem of symplectic isotopies [3].

Proposition 2.1. *Every $(U, \mathcal{H}) \in \mathfrak{T}(M, \omega, g)$ decomposes in a unique way as*

$$(2.4) \quad (U, \mathcal{H}) = (0, \mathcal{H}) \bowtie (U \circ \phi_{(0, \mathcal{H})}, 0)$$

Proof. Let $\{\phi_t\}$ be the symplectic isotopy generated by (U, \mathcal{H}) . It follows from [3] that $\{\phi_t\}$ decomposes in a unique way as $\{\phi_t\} = \{\rho_t\} \circ \{\psi_t\}$ where $\{\rho_t\}$ is a harmonic isotopy and $\{\psi_t\}$ is a Hamiltonian isotopy. Now, for each t , we compute $\dot{\phi}_t = \dot{\rho}_t + (\rho_t)_*(\dot{\psi}_t)$ and derive that

$$\iota(\dot{\phi}_t)\omega = \iota(\dot{\rho}_t)\omega + (\rho_t^{-1})^*(\iota(\dot{\psi}_t)\omega) = \mathcal{H}_t + dU_t.$$

It follows immediately from the above identities that $\iota(\dot{\rho}_t)\omega = \mathcal{H}_t$ and $\iota(\dot{\psi}_t)\omega = (\rho_t)^*(dU_t)$ for each t , i.e. $\{\rho_t\}$ is generated by $(0, \mathcal{H})$ and $\{\psi_t\}$ is generated by $(U \circ \{\rho_t\}, 0)$ so that $(U, \mathcal{H}) = (0, \mathcal{H}) \bowtie (U \circ \{\rho_t\}, 0)$. The uniqueness of this decomposition is supported by the uniqueness of the Hodge decomposition of symplectic isotopies [3]. This completes the proof. \square

2.5. Reparameterization of symplectic isotopies [2, 10]. We shall need the following basic formula for the generator of a reparameterized symplectic path. Let $\Phi = \{\phi_t\} \in Iso(M, \omega)$ which is generated by (U, \mathcal{H}) , and let $\xi : [0, 1] \rightarrow [0, 1]$ be a smooth function. The reparameterized path $\Phi^\xi : t \mapsto \phi_{\xi(t)}$ is generated by the element $(U, \mathcal{H})^\xi$ defined by

$$(2.5) \quad (U, \mathcal{H})^\xi = (U^\xi, \mathcal{H}^\xi),$$

where \mathcal{H}^ξ is the smooth map $t \mapsto \dot{\xi}(t)\mathcal{H}_{\xi(t)}$, while U^ξ is the smooth map $t \mapsto \dot{\xi}(t)U_{\xi(t)}$, and $\dot{\xi}(t)$ is the derivative of ξ with respect to t .

Definition 2.2. ([10]). Given a curve $\xi : [0, 1] \rightarrow \mathbb{R}$, its norm $\|\xi\|_{ham}$ is defined by $\|\xi\|_{ham} = \|\xi\|_{C^0} + \|\dot{\xi}\|_{L_1}$ where $\|\dot{\xi}\|_{L_1} = \int_0^1 |\dot{\xi}(t)| dt$, and $\|\xi\|_{C^0} = \sup_t |\xi(t)|$.

2.6. Boundary flat symplectic isotopies [2, 10].

Definition 2.3. ([2]). Let $(U, \mathcal{H}) \in \mathfrak{T}(M, \omega, g)$. We say that (U, \mathcal{H}) is boundary flat if there exists $\delta \in]0, 1[$ such that $(U_t, \mathcal{H}_t) = (0, 0)$ for all $t \in [0, \delta \cup]1 - \delta, 1]$.

It follows from the definition above that a path $\{\phi_t\}$ is boundary flat if there exists a constant $0 < \delta < 1$ such that $\phi_t = id$ for all $0 \leq t < \delta$ and $\phi_t = \phi_1$ for all $1 - \delta < t \leq 1$.

The following results show some properties of the Hamiltonians $\Delta(\mathcal{H}, \Phi)$.

Proposition 2.4. *Let (M, ω) be a closed symplectic manifold. If $\Phi = \{\phi_t\} \in Iso(M, \omega)$ is Hamiltonian, then for any $\mathcal{H} \in \mathcal{PH}^1(M, g)$, the Hamiltonian $\Delta(\mathcal{H}, \Phi)$ is normalized, i.e. $\int_M \Delta_t(\mathcal{H}, \Phi)\omega^n = 0$ for each t .*

Proof. Assume that $\Phi = \{\phi_t\}$ is Hamiltonian and for each t , set $Z_t = \frac{d\phi_t}{dt} \circ \phi_t^{-1}$. We use the expression $\Delta_t(\mathcal{H}, \Phi) = \int_0^t \mathcal{H}_s(Z_s) \circ \phi_s ds$ to obtain

$$(2.6) \quad \int_M \Delta_t(\mathcal{H}, \Phi)\omega^n = \int_0^t \left(\int_M \mathcal{H}_s(Z_s)\omega^n \right) ds,$$

for each t , but the identity $\mathcal{H}_t \wedge \omega^n = 0$ yields $(\iota(Z_s)\mathcal{H}_t) \wedge \omega^n + (\iota(Z_s)\omega^n) \wedge \mathcal{H}_t = 0$ for all $s \in [0, t]$, and this implies in turn that

$$(2.7) \quad \int_M \mathcal{H}_t(Z_s)\omega^n = - \int_M (\iota(Z_s)\omega^n) \wedge \mathcal{H}_t = -n \int_M d[U_s \circ \phi_s^{-1}\omega^{n-1} \wedge \mathcal{H}_t],$$

for all $s \in [0, t]$, where U denotes the generating Hamiltonian of the path Φ . Combining (2.6) and (2.7) yield,

$$(2.8) \quad \int_M \Delta_t(\mathcal{H}, \Phi)\omega^n = \int_0^t \left(\int_M \mathcal{H}_t(Z_s)\omega^n \right) ds = -n \int_M d\left[\left(\int_0^t U_s \circ \phi_s^{-1} ds \right) \omega^{n-1} \wedge \mathcal{H}_t \right],$$

Applying Stokes' theorem in the right-hand side of (2.8) leads to

$$(2.9) \quad \int_M \Delta_t(\mathcal{H}, \Phi)\omega^n = -n \int_{\partial M} \left(\int_0^t U_s \circ \phi_s^{-1} ds \right) \omega^{n-1} \wedge \mathcal{H}_t = 0$$

since $\partial M = \emptyset$. This completes the proof. \square

Lemma 2.5. *Let $\Phi_1 = (\phi_1^t)$ and $\Phi_2 = (\phi_2^t)$ be two symplectic isotopies. Let \mathcal{H} be a smooth family of closed 1-forms. Then, for each t , there exists a constant C which depends on Φ_1 , Φ_2 , and t such that,*

$$\Delta_t(\mathcal{H}, \Phi_1 \circ \Phi_2) = \Delta_t(\mathcal{H}, \Phi_2) + \Delta_t(\mathcal{H}, \Phi_1) \circ \phi_2^t + C.$$

Proof. For a fixed t , we have,

$$\begin{aligned} d\Delta_t(\mathcal{H}, \Phi_1 \circ \Phi_2) &= (\phi_1^t \circ \phi_2^t)^* \mathcal{H}_t - \mathcal{H}_t \\ &= (\phi_2^t)^* ((\phi_1^t)^* (\mathcal{H}_t)) - \mathcal{H}_t \\ &= (\phi_2^t)^* (\mathcal{H}_t + d\Delta_t(\mathcal{H}, \Phi_1)) - \mathcal{H}_t \\ &= (\phi_2^t)^* (\mathcal{H}_t) + d\Delta_t(\mathcal{H}, \Phi_1) \circ \phi_2^t - \mathcal{H}_t \\ &= \mathcal{H}_t + d\Delta_t(\mathcal{H}, \Phi_2) + d\Delta_t(\mathcal{H}, \Phi_1) \circ \phi_2^t - \mathcal{H}_t \\ &= d\Delta_t(\mathcal{H}, \Phi_2) + d\Delta_t(\mathcal{H}, \Phi_1) \circ \phi_2^t. \end{aligned}$$

It follows from the above estimates that :

$$\Delta_t(\mathcal{H}, \Phi_1 \circ \Phi_2) = \Delta_t(\mathcal{H}, \Phi_2) + \Delta_t(\mathcal{H}, \Phi_1) \circ \Phi_2 + C.$$

This achieves the proof. \square

2.7. The C^0 -metric. Let $Homeo(M)$ be the homeomorphisms' group of M equipped with the C^0 -compact-open topology. This is the metric topology induced by the distance

$$d_0(f, h) = \max(d_{C^0}(f, h), d_{C^0}(f^{-1}, h^{-1})),$$

where $d_{C^0}(f, h) = \sup_{x \in M} d(h(x), f(x))$ and d is a distance on M induced by the Riemannian metric g . On the space of all continuous paths $\varrho : [0, 1] \rightarrow Homeo(M)$ such that $\varrho(0) = id$, we consider the C^0 -topology as the metric topology induced by the metric

$$\bar{d}(\lambda, \mu) = \max_{t \in [0, 1]} d_0(\lambda(t), \mu(t)).$$

2.8. Scholium. Here, using Hopf-Rinow theorem, we show some bounded properties of the Hamiltonians $\Delta(\mathcal{H}, \Phi)$ with respect to the Hofer-norms. This will be useful later on.

When the manifold M admits a complete Riemannian metric, by Hopf-Rinow theorem one can choose the path γ to be a geodesic, and its length is bounded from above by the diameter of the manifold M , i.e. $\int_0^1 \|\dot{\gamma}(s)\|_g ds \leq \text{diam}(M)$ where $\text{diam}(M)$ denotes the diameter of M with respect to the Riemannian metric g . Let $\mathcal{H} = (\mathcal{H}_t) \in \mathcal{PH}^1(M, g)$ and $\{\phi_t\}$ be an isotopy. Since for each t , the harmonic 1-form \mathcal{H}_t is closed, it follows that

$$(2.10) \quad \phi_t^*(\mathcal{H}_t) - \mathcal{H}_t = d\Delta_t(\mathcal{H}, \{\phi_t\}),$$

for each t . It follows from (2.10) that

$$(2.11) \quad u_t(x) := \int_{\gamma} \phi_t^*(\mathcal{H}_t) - \mathcal{H}_t = \Delta_t(\mathcal{H}, \{\phi_t\})(x) - \Delta_t(\mathcal{H}, \{\phi_t\})(x_0),$$

for each t . In view of the above fact, to study the behavior of the Hamiltonian $(x, t) \mapsto \Delta_t(\mathcal{H}, \{\phi_t\})(x)$ with respect to the Hofer topologies we will only need to study the behavior of its associated Hamiltonian $(x, t) \mapsto u_t(x)$ with respect to the uniform sup norm since the two norms are equivalent. For instance, let y_0 be any point of M that realizes the supremum of the function $x \mapsto |u_t(x)|$. We derive from the triangle inequality that,

$$\begin{aligned} \sup_x |u_t(x)| &\leq \left| \int_{\gamma_{y_0}} \mathcal{H}_t \right| + \left| \int_{\gamma_{y_0}} \phi_t^*(\mathcal{H}_t) \right| \\ &\leq |\mathcal{H}_t| \int_0^1 \|\dot{\gamma}_{y_0}(s)\|_g ds + \sup_{t,s} |D\phi_t(\gamma_{y_0}(s))| |\mathcal{H}_t| \int_0^1 \|\dot{\gamma}_{y_0}(s)\|_g ds, \\ &\leq \text{diam}(M) (1 + \sup_{t,s} |D\phi_t(\gamma_{y_0}(s))|) |\mathcal{H}_t|, \end{aligned}$$

where $D\phi_t$ is the tangent map of ϕ_t . This yields the following estimates

$$(2.12) \quad \int_0^1 \text{osc}(\Delta_t(\mathcal{H}, \{\phi_t\})) dt \leq 2\text{diam}(M) (1 + \sup_{t,s} |D\phi_t(\gamma_{y_0}(s))|) \int_0^1 |\mathcal{H}_t| dt.$$

$$(2.13) \quad \max_{t \in [0,1]} \text{osc}(\Delta_t(\mathcal{H}, \{\phi_t\})) \leq 2\text{diam}(M) (1 + \sup_{t,s} |D\phi_t(\gamma_{y_0}(s))|) \max_{t \in [0,1]} |\mathcal{H}_t|.$$

2.9. Regularization of symplectic isotopies. A regularization process for Hamiltonian paths is due to Polterovich [11]. As far as we know a general regularization process for symplectic isotopies is unknown. The pathen is underlying to the proof of Lemma 3.3 found in [2]. Here, we use Hodge's theory to show that Polterovich's regularization process for Hamiltonian isotopies admits a natural generalization to symplectic isotopies. Before we start, note that a symplectic path $\{\phi_t\}$ is said to be regular if for every t , the tangent vector $\dot{\phi}_t$ to the path $\{\phi_t\}$ does not vanish. For instance, let $\Phi = \phi_{(U, \mathcal{H})} \in \text{Iso}(M, \omega)$, in view of a Polterovich's result ([11], Proposition 5.2.A), for the above Hamiltonian U , there exists a Hamiltonian loop $\phi_{(r,0)}$ which is close to the constant loop identity (in the C^∞ -sense), and in particular, its generating function r is arbitrarily small in the $L^{(1,\infty)}$ version of Hofer's norm so that

$$(2.14) \quad \text{osc}(-r_t + U_t) \neq 0$$

for all t . Now, consider (V, \mathcal{K}) to be the product $\overline{(r, 0)} \rtimes (U, \mathcal{H})$, which can be written immediately as

$$(2.15) \quad (V, \mathcal{K}) = (-r \circ \phi_{(r,0)} + U \circ \phi_{(r,0)} + \tilde{\Delta}(\mathcal{H}, \phi_{(r,0)}), \mathcal{H}).$$

We claim that the isotopy generated by the element (V, \mathcal{K}) just constructed above is regular. As a matter of fact, assume that there exists a time s for which the vector field $X_s = \dot{\phi}_{(V, \mathcal{K})}^s$ vanishes identically, and this is equivalent to,

$$(2.16) \quad \iota(X_s)\omega = dV_s + \mathcal{K}_s = 0.$$

Inserting (2.15) in (2.16), we obtain

$$(2.17) \quad d(-r_s \circ \phi_{(r,0)}^s + U_s \circ \phi_{(r,0)}^s) + \Delta_s(\mathcal{H}, \phi_{(r,0)}) + \mathcal{H}_s = 0.$$

From (2.17), it follows that the harmonic 1-form \mathcal{H}_s is exact, and then the latter must be trivial because in view of Hodge's theory [13], any exact harmonic form is trivial. This suggests that $\Delta_s(\mathcal{H}, \phi_{(r,0)}) = 0$, which implies in turn that the function $x \mapsto (-r_s \circ \phi_{(r,0)}^s + U_s \circ \phi_{(r,0)}^s)(x)$ is constant. This contradicts the assumption $\text{osc}(r_t - U_t) \neq 0$ for all t , and the claim follows.

As a consequence of the above regularization process, we derive that using any regular symplectic path $\phi_{(V, \mathcal{K})}$ we can define a function $\zeta : [0, 1] \rightarrow [0, 1]$ to be the inverse of the map,

$$s \mapsto \frac{\int_0^s (\text{osc}(V_t) + |\mathcal{K}_t|) dt}{\int_0^1 (\text{osc}(V_t) + |\mathcal{K}_t|) dt},$$

and the derivative of ζ is given explicitly by :

$$(2.18) \quad \zeta'(s) = \frac{\int_0^1 (\text{osc}(V_t) + |\mathcal{K}_t|) dt}{\text{osc}(V_{\zeta(s)}) + |\mathcal{K}_{\zeta(s)}|}.$$

Note that if ζ is only C^1 , then we can approximate ζ in the C^1 -topology by a smooth diffeomorphisms $\kappa : [0, 1] \rightarrow [0, 1]$ that fixes 0 and 1 (see [6]). This will enable us to prove the uniqueness of Banyaga's Hofer-like metric.

3. MAIN RESULTS

Throughout this section, we introduce the main results of this paper.

We shall need the definitions of Banyaga's topologies [3].

According to [3], the so-called $L^{(1, \infty)}$ version and L^∞ version of Banyaga's Hofer-like lengths of any $\Phi = \phi_{(U, \mathcal{H})} \in \text{Iso}(M, \omega)$ are defined respectively by,

$$(3.1) \quad l^{(1, \infty)}(\Phi) = \int_0^1 \text{osc}(U_t) + |\mathcal{H}_t| dt,$$

$$(3.2) \quad l^\infty(\Phi) = \max_{t \in [0, 1]} (\text{osc}(U_t) + |\mathcal{H}_t|).$$

Clearly $l^{(1, \infty)}(\Phi) \neq l^{(1, \infty)}(\Phi^{-1})$ unless Φ is Hamiltonian. Indeed, $\Phi = \phi_{(U, \mathcal{H})}$ implies that $\Phi^{-1} = \phi_{(\overline{U}, \overline{\mathcal{H}})}$ where $(\overline{U}, \overline{\mathcal{H}}) = (-U \circ \Phi - \tilde{\Delta}(\mathcal{H}, \Phi), -\mathcal{H})$. Hence, we see that the mean oscillation of the function U can be different from that of the function $-U \circ \Phi - \tilde{\Delta}(\mathcal{H}, \Phi)$. But, if Φ is Hamiltonian, i.e. $\Phi = \phi_{(U, 0)}$, then the mean oscillation of U is equal to that of $-U \circ \Phi$, i.e. $l^{(1, \infty)}(\Phi) = l^{(1, \infty)}(\Phi^{-1})$. Similarly, we have $l^\infty(\Phi) \neq l^\infty(\Phi^{-1})$ unless Φ is Hamiltonian.

Before we start, let's revisit an interesting result from Hamiltonian dynamics. In the special case that $\text{Ham}(M, \omega) = \text{Symp}_0(M, \omega)$ (or $H^1(M, \mathbb{R}) = 0$), it is known that the

Hofer length for Hamiltonian paths have an impact in the investigation of the Hamiltonian nature of a homeomorphism which is the C^0 -limit of a sequence of Hamiltonian diffeomorphisms ([8], Theorem 6).

When $H^1(M, \mathbb{R}) \neq 0$, analyzing the proof given by Hofer-Zehnder [8], it turns out that in the presence of the positivity result of the symplectic displacement energy, one can prove a symplectic analogue of Theorem 6 found by Hofer-Zehnder [8] (in the context of Banyaga's Hofer-like geometry). This shall follow closely to the proof given by Hofer-Zehnder [8]. The above arguments lead to the following delicate question. In the lack of the positivity result of the symplectic displacement energy, does it make sense to think of the symplectic analogue of Theorem 6 found by Hofer-Zehnder [8] in the context of Banyaga's Hofer-like geometry (when $H^1(M, \mathbb{R}) \neq 0$)?

One main theorem of this paper gives an affirmative answer to the above question. More precisely, without appealing to the positivity of a symplectic displacement energy, we prove the following main theorem.

Theorem 3.1. *Let (M, ω) be a closed symplectic manifold. Let $\Phi_i = \{\phi_i^t\}$ be a sequence of symplectic isotopies, $\Psi = \{\psi^t\} \in \text{Iso}(M, \omega)$, and $\phi : M \rightarrow M$ be a map, such that*

- (ϕ_i^1) converges uniformly to ϕ , and
- $l^\infty(\Phi_i^{-1} \circ \Psi) \rightarrow 0, i \rightarrow \infty$.

Then we must have $\phi = \psi^1$.

This result shows not only an advantage of the L^∞ Hofer-like length over the $L^{(1, \infty)}$ Hofer-like length but it yields the symplectic analogue of Theorem 6 found by Hofer-Zehnder [8] (in the L^∞ context). In fact, the choice of the L^∞ Hofer-like length is supported by the following results.

Lemma 3.2. *Let ρ_i be a sequence of harmonic isotopies generated by $(0, \mathcal{H}^i)$ and let ρ be another harmonic isotopy generated by $(0, \mathcal{H})$ such that $\max_{t \in [0, 1]} |\mathcal{H}_t^i - \mathcal{H}_t| \rightarrow 0, i \rightarrow \infty$. Then the following properties hold*

- (1) $l^\infty(\rho_i^{-1} \circ \rho) \rightarrow 0, i \rightarrow \infty$,
- (2) ρ_i converges in \bar{d} to ρ .

Proof. For (2), we define a sequence (Z_t^i) of smooth family of harmonic vector fields by setting $\iota(Z_t^i)\omega = \mathcal{H}_t^i$ for each i and for all t . Similarly, we define a smooth family (Z_t) of harmonic vector fields by setting $\iota(Z_t)\omega = \mathcal{H}_t$ for all t . Since by assumption we have

$$\max_{t \in [0, 1]} |\mathcal{H}_t^i - \mathcal{H}_t| \rightarrow 0, i \rightarrow \infty,$$

it turns out that the sequence (Z_t^i) converges uniformly to (Z_t) . Therefore, it follows from the standard continuity theorem of ODE for Lipschitz vector fields that the sequence of paths generated by (Z_t^i) must converge uniformly to the path generated by (Z_t) , i.e. ρ_i converges uniformly to ρ . For (1), we compute

$$\overline{(0, \mathcal{H}^i)} \bowtie (0, \mathcal{H}) = (\tilde{\Delta}(\mathcal{H} - \mathcal{H}^i, \rho_i), \mathcal{H} - \mathcal{H}^i),$$

for each i , and we derive that to complete the proof, we only need to prove that

$$\max_{t \in [0, 1]} \text{osc}(\tilde{\Delta}_t(\mathcal{H} - \mathcal{H}^i, \rho_i)) \rightarrow 0, i \rightarrow \infty.$$

For this purpose, we use (2.13) to derive that

$$\max_{t \in [0, 1]} \text{osc}(\Delta_t(\mathcal{H} - \mathcal{H}^i, \rho_i)) \leq 2\text{diam}(M)(1 + \sup_{t, s} |D\rho_i^t(\gamma_{y_0}(s))|) \max_{t \in [0, 1]} |\mathcal{H}_t - \mathcal{H}_t^i|,$$

where $D\rho_i^t$ stands for the tangent map of ρ_i^t for each i , $y_0 \in M$ and γ_{y_0} is a geodesic such that $\gamma_{y_0}(1) = y_0$ (see Section 2.8 of the present paper). The right-hand side in the above estimate tends to zero when i goes to the infinity since the quantity $(1 + \sup_{t,s} |D\rho_i^t(\gamma_{y_0}(s))|)$ is bounded for each i , and by assumption we have $\max_{t \in [0,1]} |\mathcal{H}_t^i - \mathcal{H}_t| \rightarrow 0, i \rightarrow \infty$. This completes the proof. \square

Remark 3.3. From Lemma 3.2, it is clear that if ρ_i is a sequence of harmonic isotopies generated by $(0, \mathcal{H}^i)$ and ρ another harmonic isotopy generated by $(0, \mathcal{H})$, then the convergence $l^\infty(\rho_i^{-1} \circ \rho) \rightarrow 0, i \rightarrow \infty$ is equivalent to the convergence $\max_{t \in [0,1]} |\mathcal{H}_t^i - \mathcal{H}_t| \rightarrow 0, i \rightarrow \infty$.

The following result shows another interesting advantage of the L^∞ Hofer-like metric.

Corollary 3.4. *Let Φ_i be a sequence of symplectic isotopies and let Ψ be a symplectic isotopy such that $l^\infty(\Phi_i^{-1} \circ \Psi) \rightarrow 0, i \rightarrow \infty$. If μ_i is the sequence of Hamiltonian paths arising in the Hodge decomposition of Φ_i , and μ the Hamiltonian path arising in the Hodge decomposition of Ψ , then $l^\infty(\mu_i^{-1} \circ \mu) \rightarrow 0, i \rightarrow \infty$.*

Proof. Assume that for each i , Φ_i is generated by (U^i, \mathcal{H}^i) and Ψ generated by (U, \mathcal{H}) . In view of Proposition 2.1, we have to prove that $\max_t \text{osc}(U_t^i \circ \phi_{(0, \mathcal{H}^i)}^t - U_t \circ \phi_{(0, \mathcal{H})}^t) \rightarrow 0, i \rightarrow \infty$. For each i , we compute

$$\text{osc}(U_t^i \circ \phi_{(0, \mathcal{H}^i)}^t - U_t \circ \phi_{(0, \mathcal{H})}^t) \leq \text{osc}(U_t^i - U_t) + \text{osc}(U_t \circ \phi_{(0, \mathcal{H}^i)}^t - U_t \circ \phi_{(0, \mathcal{H})}^t),$$

for all t , but by assumption we have $\max_t (\text{osc}(U_t^i - U_t)) \rightarrow 0, i \rightarrow \infty$, while the uniform continuity of the map $(t, x) \mapsto U_t(x)$ together with Lemma 3.2 yield

$$\max_t (\text{osc}(U_t \circ \phi_{(0, \mathcal{H}^i)}^t - U_t \circ \phi_{(0, \mathcal{H})}^t)) \rightarrow 0, i \rightarrow \infty.$$

This completes the proof. \square

Proof of Theorem 3.1. Since we use not the positivity result of the symplectic displacement energy but the Hodge decomposition theorem of symplectic isotopies, the proof of this main result is rather delicate. We shall proceed in three steps.

- **Step (1). (Convergence of symplectic isotopies).** For each i , let $\rho_i = \{\rho_i^t\}$ and $\rho = \{\rho^t\}$ denoting respectively the harmonic isotopies arising in the Hodge decompositions of the paths $\Phi_i = \{\phi_i^t\}$ and $\Psi = \{\psi^t\}$. Since by assumption, we have $l^\infty(\Phi_i^{-1} \circ \Psi) \rightarrow 0, i \rightarrow \infty$, it turns out that $l^\infty(\rho_i^{-1} \circ \rho) \rightarrow 0, i \rightarrow \infty$. This together with Lemma 3.2-(2) (or Remark 3.3) tell us that the sequence ρ_i converges in \bar{d} to ρ .
- **Step (2). (Decomposition of the map $\phi = \lim_{C^0}(\phi_i^1)$).** For each i , let $\mu_i = \{\mu_i^t\}$ and $\mu = \{\mu^t\}$ denoting respectively the Hamiltonian isotopies arising in the Hodge decompositions of the paths $\Phi_i = \{\phi_i^t\}$ and $\Psi = \{\psi^t\}$. By assumption, the sequence of time-one maps ϕ_i^1 converges uniformly to ϕ , and in view of step (1) the sequence of time-one maps ρ_i^1 converges uniformly to the time-one map ρ^1 . The preceding arguments suggest that the sequence of time-one maps μ_i^1 converges uniformly to a homeomorphism σ since $\mu_i^1 = (\rho_i^1)^{-1} \circ \phi_i^1$ for each i . For instance, we compute

$$d_{C^0}(\phi, \rho^1 \circ \sigma) \leq d_{C^0}(\phi, \rho_i^1 \circ \mu_i^1) + d_{C^0}(\rho_i^1 \circ \mu_i^1, \rho_i^1 \circ \sigma) + d_{C^0}(\rho_i^1 \circ \sigma, \rho^1 \circ \sigma),$$

for each i , and derive that $\phi = \rho^1 \circ \sigma$ since by assumption, $d_{C^0}(\phi, \rho_i^1 \circ \mu_i^1) = d_{C^0}(\phi, \phi_i^1) \rightarrow 0, i \rightarrow \infty$, and from the bi-invariance of the metric d_{C^0} it follows

that

$$\begin{aligned} d_{C^0}(\rho_i^1 \circ \mu_i^1, \rho_i^1 \circ \sigma) &= d_{C^0}(\mu_i^1, \sigma) \rightarrow 0, i \rightarrow \infty, \\ d_{C^0}(\rho_i^1 \circ \sigma, \rho^1 \circ \sigma) &= d_{C^0}(\rho_i^1, \rho^1) \rightarrow 0, i \rightarrow \infty. \end{aligned}$$

- **Step (3). (The Hamiltonian nature of the map σ).** To achieve the proof, all we have to show is that $\sigma = \mu^1$ where μ^1 is the time-one map of the Hamiltonian path $\mu = \{\mu^t\}$. Arguing indirectly, we find that there exists a small non-empty closed ball $B \subset M$ which is completely displaced by $\sigma^{-1} \circ \mu^1$, i.e. $B \cap [\sigma^{-1} \circ \mu^1](B) = \emptyset$. Since B is compact and the convergence $\mu_i^1 \rightarrow \sigma$ is uniform, we must have $B \cap [(\mu_i^1)^{-1} \circ \mu^1](B) = \emptyset$ for all sufficiently large i . The above arguments tell us that we can apply the energy-capacity inequality theorem from [9] to obtain

$$(3.3) \quad 0 < C(B)/2 \leq l^\infty(\mu_i^{-1} \circ \mu),$$

for all sufficiently large i , where $C(B)$ represents the Gromov area of the ball B . But in view of Corollary 3.4, the right-hand side in (3.3) tends to zero when i goes to the infinity. This contradicts the assumption $0 < C(B)/2$. Therefore, $\sigma = \mu^1$, and this yields $\phi = \rho^1 \circ \mu^1 = \psi^1$. This completes the proof.

The following result is an immediate consequence of Theorem 3.1. It can justify the definition of strong symplectic isotopies in the L^∞ context [2, 4, 12].

Corollary 3.5. *Let (M, ω) be a closed symplectic manifold. Let $\Phi_i = \{\phi_i^t\}$ be a sequence of symplectic isotopies, let $\Psi = \{\psi_t\} \in \text{Iso}(M, \omega)$, and $\eta : t \mapsto \eta_t$ a family of maps $\eta_t : M \rightarrow M$, such that Φ_i converges in \bar{d} to η and $l^\infty(\Phi_i^{-1} \circ \Psi) \rightarrow 0, i \rightarrow \infty$. Then $\eta = \Psi$.*

Proof. Assume the contrary that $\Psi \neq \eta$, i.e. there exists $t \in]0, 1]$ such that $\eta_t \neq \psi_t$. Then the sequence of symplectic paths $\Xi_i : s \mapsto \phi_i^{st}$ contradicts Theorem 3.1. This completes the proof. \square

3.1. Banyaga's Hofer-like norms. Before we continue with further investigation of Banyaga topologies, we shall need the following notions.

Let $\phi \in \text{Symp}_0(M, \omega)$, using the above Banyaga's lengths introduced above, Banyaga [3] defined respectively the $L^{(1, \infty)}$ energy and L^∞ energy of ϕ by,

$$(3.4) \quad e_0(\phi) = \inf(l^{(1, \infty)}(\Phi)),$$

$$(3.5) \quad e_0^\infty(\phi) = \inf(l^\infty(\Phi)),$$

where the infimum are taken over all symplectic isotopies Φ with time-one map equal to ϕ . Therefore, the $L^{(1, \infty)}$ Banyaga's Hofer-like norm and the L^∞ Banyaga's Hofer-like norm of ϕ are respectively defined by,

$$(3.6) \quad \|\phi\|_{HL}^{(1, \infty)} = (e_0(\phi) + e_0(\phi^{-1}))/2,$$

$$(3.7) \quad \|\phi\|_{HL}^\infty = (e_0^\infty(\phi) + e_0^\infty(\phi^{-1}))/2.$$

Each of the norms $\|\cdot\|_{HL}^\infty$ and $\|\cdot\|_{HL}^{(1, \infty)}$ generalizes the Hofer norms for Hamiltonian diffeomorphisms in the following sense : in the special case of a closed symplectic manifold (M, ω) for which $\text{Ham}(M, \omega) = \text{Symp}_0(M, \omega)$ (or $H^1(M, \mathbb{R}) = 0$), the norm $\|\cdot\|_{HL}^\infty$ reduces to a norm $\|\cdot\|_H^\infty$ called the L^∞ Hofer norm, while the norm $\|\cdot\|_{HL}^{(1, \infty)}$ reduces to a norm $\|\cdot\|_H^{(1, \infty)}$ called the $L^{(1, \infty)}$ Hofer norm. But, a result that was proved by Polterovich [11] shows that the above Hofer's norms are equal in general, i.e. $\|\cdot\|_H^{(1, \infty)} = \|\cdot\|_H^\infty$. In

other words the norms $\|\cdot\|_{HL}^\infty$ and $\|\cdot\|_{HL}^{(1,\infty)}$ are equal when $Ham(M, \omega) = Symp_0(M, \omega)$ (or $H^1(M, \mathbb{R}) = 0$). However, when $Symp_0(M, \omega) \setminus Ham(M, \omega) \neq \emptyset$ (or $H^1(M, \mathbb{R}) \neq 0$), it is unknown whether the norms $\|\cdot\|_{HL}^\infty$ and $\|\cdot\|_{HL}^{(1,\infty)}$ are equal or not. This motivated the following main lemma.

Lemma 3.6. *Let (M, ω) be a closed symplectic manifold. For every $\phi \in Symp_0(M, \omega)$, we have*

$$\|\phi\|_{HL}^\infty = \|\phi\|_{HL}^{(1,\infty)}.$$

This result was announced in Banyaga-Tchuiaga [2] without any explicit proof. It yields the symplectic analogue of a result which is due to Polterovich ([11], Lemma 5.1.C). Its proof is based on the following lemma which is a refined version of Lemma 3.3 found in [2].

Lemma 3.7. *Let (M, ω) be a closed symplectic manifold. Let Φ be a symplectic isotopy, and let ϵ be a positive real number. Then, there exists Ψ be a symplectic isotopy with the same extremities than Φ which is regular such that $l^\infty(\Psi) < l^{(1,\infty)}(\Phi) + \epsilon$.*

Proof of Lemma 3.6. The inequality $\|\cdot\|_{HL}^{(1,\infty)} \leq \|\cdot\|_{HL}^\infty$ is clear from the definition of the energies. For the converse, let $\phi \in Symp_0(M, \omega)$, we derive from the characterization of the infimum that for all positive real number ϵ , we can find a symplectic path Φ_ϵ that connects ϕ to the identity such that $l^{(1,\infty)}(\Phi_\epsilon) \leq e_0(\phi) + \epsilon$. But, Lemma 3.7 shows that there exists $\Psi_\epsilon \in Iso(M, \omega)$ with the same extremities than Φ_ϵ such that $l^\infty(\Psi_\epsilon) < l^{(1,\infty)}(\Phi_\epsilon) + \epsilon$. This yields $e_0^\infty(\phi) \leq l^\infty(\Psi_\epsilon) < e_0(\phi) + 2\epsilon$, i.e. $e_0^\infty(\phi) < e_0(\phi) + 2\epsilon$. Similarly, we use Lemma 3.7 to derive that $e_0^\infty(\phi^{-1}) < e_0(\phi^{-1}) + 2\epsilon$. Therefore, we summarize the above estimates to get

$$(3.8) \quad \|\phi\|_{HL}^\infty = (e_0^\infty(\phi^{-1}) + e_0^\infty(\phi))/2 < (e_0(\phi) + e_0(\phi^{-1}) + 4\epsilon)/2 \leq \|\phi\|_{HL}^{(1,\infty)} + 2\epsilon.$$

Since (3.8) holds for all arbitrary positive ϵ , we conclude that $\|\phi\|_{HL}^\infty \leq \|\phi\|_{HL}^{(1,\infty)}$. This completes the proof. \square

3.2. Proof of Lemma 3.7. In the rest of this paper, we will always denote by $r(g)$, the injectivity radius of a Riemannian metric g . We will need the following result.

Lemma 3.8. *Let (M, g) be a closed oriented Riemannian manifold. Let $\mathcal{H} \in \mathcal{PH}^1(M, g)$. The following facts hold :*

- (1) *Let $\Psi = \{\psi_t\}$ be an isotopy, and let $\xi_1, \xi_2 : [0, 1] \rightarrow [0, 1]$ be two smooth monotone functions that fix 0. Then there exists a constant B_2 which depends on \mathcal{H} and Ψ such that,*

$$\int_0^1 \text{osc}(\Delta_t(\mathcal{H}, \Psi^{\xi_1}) - \Delta_t(\mathcal{H}, \Psi^{\xi_2})) dt \leq B_2 \|\xi_1 - \xi_2\|_{ham}.$$

- (2) *Let $\Phi = \{\phi_t\}$ and $\Psi = \{\psi_t\}$ be two isotopies such that $\bar{d}(\Phi, \Psi) \leq r(g)/2$ where $r(g)$ is the injectivity radius of the Riemannian metric g on M . Then,*

$$\int_0^1 \text{osc}(\Delta_t(\mathcal{H}, \Phi) - \Delta_t(\mathcal{H}, \Psi)) dt \leq 4 \max_t |\mathcal{H}_t| \bar{d}(\Phi, \Psi).$$

Proof. For each $j = 1, 2$, differentiating the reparameterized path Ψ^{ξ_j} in the variable t yields $\dot{\Psi}^{\xi_j}(t) = \dot{\xi}_j(t)\dot{\psi}_{\xi_j(t)}$ for all $t \in [0, 1]$. Compute

$$\Delta_t(\mathcal{H}, \Psi^{\xi_j}) = \int_0^t \mathcal{H}_t(\dot{\Psi}^{\xi_j}(s)) \circ \Psi^{\xi_j}(s) ds = \int_0^t \dot{\xi}_j(s) \mathcal{H}_t(\dot{\psi}_{\xi_j(s)}) \circ \psi_{\xi_j(s)} ds,$$

for each t . By a suitable change of variable, the right-hand side in the above estimates is written as $\int_0^t \dot{\xi}_j(s) \mathcal{H}_t(\dot{\psi}_{\xi_j(s)}) \circ \psi_{\xi_j(s)} ds = \int_0^{\xi_j(t)} \mathcal{H}_t(\dot{\psi}_u) \circ \psi_u du$. This in turn yields

$$\begin{aligned} \int_0^1 \text{osc}(\Delta_t(\mathcal{H}, \Psi^{\xi_1}) - \Delta_t(\mathcal{H}, \Psi^{\xi_2})) dt &= \int_0^1 \text{osc}\left(\int_{\min\{\xi_1(t), \xi_2(t)\}}^{\max\{\xi_1(t), \xi_2(t)\}} \mathcal{H}_t(\dot{\psi}_u) \circ \psi_u du\right) dt \\ &\leq 2 \sup_{s,t,x} |\mathcal{H}_t(\dot{\psi}_s)(x)| \|\xi_1 - \xi_2\|_{C^0} \\ &\leq 2 \sup_{s,t,x} |\mathcal{H}_t(\dot{\psi}_s)(x)| \|\xi_1 - \xi_2\|_{ham}. \end{aligned}$$

Therefore, the desired B_2 is given by $B_2 = 2 \sup_{s,t,x} |\mathcal{H}_t(\dot{\psi}_s)(x)| < \infty$.

For (2), we set $\Phi = \{\phi_t\}$ and $\Psi = \{\psi_t\}$, and from the bi-invariance of the metric \bar{d} , we derive from the assumption that $\bar{d}(\Phi, \Psi) = \bar{d}(\Phi \circ \Psi^{-1}, Id) \leq r(g)/2$. Under this condition, it follows from the lines of proof of Lemma 3.2 found in [2] that

$$\int_0^1 \text{osc}(\Delta_t(\mathcal{H}, \Phi \circ \Psi^{-1})) dt \leq 4 \max_t |\mathcal{H}_t| \bar{d}(\Phi \circ \Psi^{-1}, Id).$$

On the other hand, we derive from Lemma 2.5 that

$$\Delta_t(\mathcal{H}, \Phi \circ \Psi^{-1}) = \Delta_t(\mathcal{H}, \Psi^{-1}) + \Delta_t(\mathcal{H}, \Phi) \circ \Psi^{-1} + cte_1,$$

and

$$0 = \Delta_t(\mathcal{H}, \Psi^{-1}) + \Delta_t(\mathcal{H}, \Psi) \circ \Psi^{-1} + cte_2.$$

That is, for each t , there exists a constant C which depends on t such that

$$\Delta_t(\mathcal{H}, \Phi \circ \Psi^{-1}) = -\Delta_t(\mathcal{H}, \Psi) \circ \Psi^{-1} + \Delta_t(\mathcal{H}, \Phi) \circ \Psi^{-1} + C,$$

i.e.

$$\begin{aligned} \int_0^1 \text{osc}(-\Delta_t(\mathcal{H}, \Psi) \circ \Psi^{-1} + \Delta_t(\mathcal{H}, \Phi) \circ \Psi^{-1}) dt &= \int_0^1 \text{osc}(\Delta_t(\mathcal{H}, \Phi \circ \Psi^{-1})) dt \\ &\leq 4 \max_t |\mathcal{H}_t| \bar{d}(\Phi \circ \Psi^{-1}, Id). \end{aligned}$$

That achieves the proof. \square

Proof of Lemma 3.7. Assume that Φ is generated by (U, \mathcal{H}) , and consider Ξ to be the path obtained by regularizing the path Φ as explained in Section 2.9. It follows that the path Ξ is generated by an element (V, \mathcal{K}) so that

$$l^{(1,\infty)}(\Xi) = \int_0^1 (\text{osc}(V_t) + |\mathcal{K}_t|) dt \leq l^{(1,\infty)}(\Phi) + \int_0^1 \text{osc}(r_t) dt + \int_0^1 \text{osc}(\Delta_t(\mathcal{H}, \phi_{(r,0)})) dt,$$

where $\phi_{(r,0)}$ is a Hamiltonian loop such that $\int_0^1 \text{osc}(r_t) dt < \epsilon/2$ (see Section 2.9 of the present paper). Since Polterovich's arguments provided in Section 2.9 state that the path $\phi_{(r,0)}$ is arbitrarily close to the constant path identity (in the C^∞ -topology), we then derive from Lemma 3.8-(2) that $\int_0^1 \text{osc}(\Delta_t(\mathcal{H}, \phi_{(r,0)})) dt < \epsilon/2$. We summarize the above statements to get $l^{(1,\infty)}(\Xi) \leq l^{(1,\infty)}(\Phi) + \epsilon$. Now, we use the path Ξ to define a curve ζ

as explained in Section 2.9 of the present paper. Let Ξ^ζ be the path obtained by a reparameterization of Ξ via ζ . For each s , set $\Omega_s = \zeta'(s)(\text{osc}(V_{\zeta(s)}) + |\mathcal{K}_{\zeta(s)}|)$, and derive from (2.18) that $\Omega_s = l^{(1,\infty)}(\Xi)$. This yields $\max_s \Omega_s = l^\infty(\Xi^\zeta) = l^{(1,\infty)}(\Xi)$. It follows from the above arguments that $l^\infty(\Xi^\zeta) = l^{(1,\infty)}(\Xi)$ and $l^{(1,\infty)}(\Xi) < l^{(1,\infty)}(\Phi) + \epsilon$, i.e. $l^\infty(\Xi^\zeta) < l^{(1,\infty)}(\Phi) + \epsilon$. Therefore, to complete the proof, it suffices to take $\Psi = \Xi^\zeta$. \square

The above proof of Lemma 3.7 is a refinement and a simplification of the proof of similar result found in [2]. This is done in view to facilitate the readers to better understand the result of this paper.

Remark 3.9. As in [10], we have $l^{(1,\infty)}(\cdot) \leq l^\infty(\cdot)$ in general, where the former is invariant under reparameterization, while the latter is far from being invariant. But, as we can read in the proof of Lemma 3.7, any regular symplectic path Φ can be reparameterized to obtain another path Ψ with the same extremities than Φ so that $l^{(1,\infty)}(\Psi) = l^\infty(\Psi)$.

3.3. Some remarks on flux geometry. Another main result of this paper deals with flux geometry.

It is known that the symplectic area swept by any smooth loop in M under the action of any Hamiltonian isotopy is null:

Theorem 3.10. (*Banyaga, [5]*). *Let (M, ω) be a closed symplectic manifold and Φ be any Hamiltonian isotopy. For any loop $\gamma \subset M$, we have $\text{Flux}(\Phi) \cdot [\gamma] = 0$.*

As far as i known, an explicit study of the dual (in a certain sense) of the latter result is not yet done. This can be formalized as follows. Given an arbitrary non-Hamiltonian isotopy, is there any property that may satisfy a non trivial loop in M so that the symplectic area swept by the latter under the isotopy in question vanishes? More generally, given any non exact closed 1-form α over M , is there any constructive method for generating non trivial smooth paths γ in M such that $\int_\gamma \alpha = 0$?

It is not too hard to see that for any given closed 1-form α , the function $\Delta_1(\alpha, \Phi)$ has a precise geometrical meaning: for each $x \in M$, where $\gamma_{x,\Phi}(t) = \Phi_t(x)$, i.e. for each $x \in M$, we can describe the real number $\Delta_1(\alpha, \Phi)(x)$ as the algebraic value of the symplectic area of the 2-chain swept by the orbit $t \mapsto \Phi_t(x)$, under the symplectic flow generated by α . It follows quite naturally from the above arguments that each zero of the function $\Delta_1(\alpha, \Phi)$ gives rise to a null symplectic area 2-chain or a solution of the equation

$$\int_\gamma \alpha = 0,$$

with unknown γ . However, we have no guarantee whether such a function always admit at least a zero or not. Here is a sufficient condition which guarantees the existence of at least one zero for such a function.

Lemma 3.11. (*Hamiltonian criterion*). *Let (M, ω) be a closed symplectic manifold. Let α be a closed 1-form over M , and Φ be a Hamiltonian isotopy. Then for each representative Ψ in the homotopic class of Φ (relatively to fix extremities) the function $x \mapsto \Delta_1(\alpha, \Psi)(x)$ has at least one zero in M .*

Proof : Since the function $x \mapsto \Delta_1(\alpha, \Phi)(x)$ is smooth and M compact, the latter function achieves its bounds. This suggests that

$$\min_{x \in M} \Delta_1(\alpha, \Phi)(x) \int_M \omega^n \leq \int_M \Delta_1(\alpha, \Phi) \omega^n \leq \max_{x \in M} \Delta_1(\alpha, \Phi)(x) \int_M \omega^n,$$

i.e.

$$\min_{x \in M} \Delta_1(\alpha, \Phi)(x) \leq 0,$$

and

$$0 \leq \max_{x \in M} \Delta_1(\alpha, \Phi)(x),$$

since Φ is Hamiltonian (see Proposition 2.4 of the present paper). On the other hand, consider the following Poincaré's scalar product :

$$\langle, \rangle_P : H^1(M, \mathbb{R}) \times H^{2n-1}(M, \mathbb{R}) \rightarrow \mathbb{R},$$

$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta,$$

where $H^*(M, \mathbb{R})$ represents the $*$ -th de Rham cohomology group. Using this bilinear mapping, one checks from the proof of Proposition 2.4 that

$$(3.9) \quad \int_M \Delta_1(\alpha, \Phi) \omega^n = n \langle Flux(\Phi), [\alpha \wedge \omega^{(n-1)}] \rangle_P$$

for all symplectic isotopy Φ , where $Flux$ represents the first Calabi's invariant. In particular, we see that the mean value $\int_M \Delta_1(\alpha, \Phi) \omega^n$ depends only on the homotopic class of Φ (relatively to fix extremities). Therefore, we get

$$\int_M \Delta_1(\alpha, \Psi) \omega^n = \int_M \Delta_1(\alpha, \Phi) \omega^n = 0.$$

This completes the proof. \square

The third main result of this paper is the following theorem.

Theorem 3.12. *Let (M, ω) be a closed symplectic manifold. Let Ψ be any symplectic isotopy whose flux is not trivial. Then any loop $\gamma \subset M$ which is homotopic to a closed Hamiltonian orbit (relatively to a fixed base point) trivializes the flux of Ψ , i.e. $Flux(\Psi) \cdot [\gamma] = 0$.*

Theorem 3.12 states that on a closed symplectic manifold, if a loop is homotopic to a closed Hamiltonian orbit (relatively to a fixed base point), then the symplectic area swept by the latter under the symplectic flow generated by any non exact closed 1-form is trivial. This can be viewed in a certain sense as the dual form of Theorem 3.10.

Proof of Theorem 3.12 : Let Ψ be a symplectic isotopy whose flux is not trivial, and denote by \mathcal{H}^Ψ the harmonic representative of the de Rham cohomology class $Flux(\Psi)$. Consider a Hamiltonian loop $\Phi = (\phi_t)$ in the fundamental group $\pi_1(Symp_0(M, \omega))$ of the group $Symp_0(M, \omega)$. In particular, since Φ is a loop we derive from Equation (2.10) that

$$d\Delta_1(\mathcal{H}^\Psi, \Phi) = 0,$$

i.e. the function $\Delta_1(\mathcal{H}^\Psi, \Phi)$ is constant. But, Lemma 3.11 suggests that the latter function must vanish since Φ is Hamiltonian. Therefore, the connectedness of M imposes that the function $\Delta_1(\mathcal{H}^\Psi, \Phi)$ must be trivial. For each $x \in M$, consider the loop $\gamma_{x, \Phi}^t = \phi_t(x)$, and check that

$$Flux(\Psi) \cdot [\gamma_{x, \Phi}] = \int_{\gamma_{x, \Phi}} \mathcal{H}^\Psi = \int_0^1 \mathcal{H}_{\gamma_{x, \Phi}^s}^\Psi (\dot{\gamma}_{x, \Phi}^s) ds = \Delta_1(\mathcal{H}^\Psi, \Phi)(x) = 0.$$

Now, let β be a representative in $[\gamma_{x,\Phi}]$, and let h_x denotes the homotopy between β and $\gamma_{x,\Phi}$. Consider the smooth 2-chain $\oplus(\beta, \gamma_{x,\Phi}) := \{h_x(s, t) : 0 \leq s, t \leq 1\}$. Since the harmonic 1-form \mathcal{H}^Ψ is closed, it follows from Stokes' theorem that

$$0 = \int_{\oplus(\beta, \gamma_{x,\Phi})} d\mathcal{H}^\Psi = \int_{\partial \oplus(\beta, \gamma_{x,\Phi})} \mathcal{H}^\Psi = \int_{\gamma_{x,\Phi}} \mathcal{H}^\Psi - \int_{\beta} \mathcal{H}^\Psi.$$

This completes the proof. \square

Theorem 3.12 states that any loop $\gamma \subset M$ which is homotopic (relatively to a fixed base point) to a closed Hamiltonian orbit satisfies $\int_\gamma \alpha = 0$, for all closed 1-form α . This seems to suggest that each Hamiltonian loop can be viewed as the trivial element in $Hom(H^1(M, \mathbb{R}), \mathbb{R})$. More generally, this tells us that there is a linear and continuous mapping

$$\tilde{\mathcal{K}}(\Phi) : H^1(M, \mathbb{R}) \rightarrow \mathbb{R}, [\alpha] \mapsto \frac{1}{n} \int_M \Delta_1(\alpha, \Phi) \omega^n,$$

for all fixed isotopy Φ . That is, $\tilde{\mathcal{K}}(\Phi)$ belongs to $Hom(H^1(M, \mathbb{R}), \mathbb{R})$ which is isomorphic to $H_1(M, \mathbb{R})$. So, there is a natural map:

$$\tilde{\mathcal{K}} : Iso(M, \omega) \rightarrow H_1(M, \mathbb{R}), \Phi \mapsto \tilde{\mathcal{K}}(\Phi).$$

This symplectic invariant looks to be very similar to the usual mass flow.

4. SOME AUXILIARY RESULTS

In this section we prove the symplectic analogues of Lemma 3.20, Lemma 3.21 and the $L^{(1,\infty)}$ -approximation found in [10].

Definition 4.1. ([3]) The $L^{(1,\infty)}$ Banyaga's topology on the space $Iso(M, \omega)$ is the metric topology induced by the following metric :

$$(4.1) \quad D^1((U, \mathcal{H}), (V, \mathcal{K})) = \frac{D_0((U, \mathcal{H}), (V, \mathcal{K})) + D_0(\overline{(U, \mathcal{H})}, \overline{(V, \mathcal{K})})}{2},$$

where

$$(4.2) \quad D_0((U, \mathcal{H}), (V, \mathcal{K})) = \int_0^1 \text{osc}(U_t - V_t) + |\mathcal{H}_t - \mathcal{K}_t| dt.$$

We will need the following lemma.

Lemma 4.2. *Let (M, g) be a closed oriented Riemannian manifold. Let $\mathcal{H} \in \mathcal{PH}^1(M, g)$, let $\Phi = \{\phi_t\}$ be an isotopy, and let $\xi_1, \xi_2 : [0, 1] \rightarrow [0, 1]$ be two smooth functions such that ξ_1 is monotonic. Then there exists a constant B_1 which depends on \mathcal{H} and Φ such that*

$$\int_0^1 \text{osc}(\Delta_t(\mathcal{H}^{\xi_1}, \Phi) - \Delta_t(\mathcal{H}^{\xi_2}, \Phi)) dt \leq B_1 \|\xi_1 - \xi_2\|_{ham}.$$

Proof. Since $\Delta(\mathcal{H}^{\xi_2}, \Phi) - \Delta(\mathcal{H}^{\xi_1}, \Phi) = \Delta_t(\mathcal{H}^{\xi_2} - \mathcal{H}^{\xi_1}, \Phi)$, we derive from (2.11) that

$$(4.3) \quad \int_0^1 \text{osc}(\Delta_t(\mathcal{H}^{\xi_2}, \Phi) - \Delta_t(\mathcal{H}^{\xi_1}, \Phi)) dt \leq 2 \text{diam}(M) (1 + \sup_{t,s} |D\phi_t(\gamma_{y_0}(s))|) \int_0^1 |\mathcal{H}_t^{\xi_1} - \mathcal{H}_t^{\xi_2}| dt,$$

where $D\phi_t$ is the tangent map of ϕ_t , y_0 a point in M , and γ_{y_0} a minimizing geodesic such that $\gamma_{y_0}(1) = y_0$ (see Section 2.8 of the present paper). Since

$$|\mathcal{H}_t^{\xi_1} - \mathcal{H}_t^{\xi_2}| \leq |\dot{\xi}_1(t) \mathcal{H}_{\xi_1(t)} - \dot{\xi}_1(t) \mathcal{H}_{\xi_2(t)}| + |\dot{\xi}_1(t) \mathcal{H}_{\xi_2(t)} - \dot{\xi}_2(t) \mathcal{H}_{\xi_2(t)}|,$$

we use the Lipschitz nature of the map $t \mapsto \mathcal{H}_t$ to derive the existence of a constant $c_0 > 0$ which depends on \mathcal{H} such that,

$$|\mathcal{H}_t^{\xi_1} - \mathcal{H}_t^{\xi_2}| \leq \max_t |\mathcal{H}_t| |\dot{\xi}_1(t) - \dot{\xi}_2(t)| + c_0 \|\xi_1 - \xi_2\|_{C^0} |\dot{\xi}_1(t)|,$$

for each t , and this yields,

$$\begin{aligned} \int_0^1 |\mathcal{H}_t^{\xi_1} - \mathcal{H}_t^{\xi_2}| dt &\leq \max_t |\mathcal{H}_t| \int_0^1 |\dot{\xi}_1(t) - \dot{\xi}_2(t)| dt + c_0 \|\xi_1 - \xi_2\|_{C^0} \\ &\leq 2 \max(c_0, \max_t |\mathcal{H}_t|) \|\xi_1 - \xi_2\|_{ham}. \end{aligned}$$

Inserting the above estimates in (4.3) yields

$$\int_0^1 \text{osc}(\Delta_t(\mathcal{H}^{\xi_2}, \Phi) - \Delta_t(\mathcal{H}^{\xi_1}, \Phi)) dt \leq B_1 \|\xi_1 - \xi_2\|_{ham},$$

where $B_1 = 4 \text{diam}(M) \max(c_0, \max_t |\mathcal{H}_t|) (1 + \sup_{t,s} |D\phi_t(\gamma_{y_0}(s))|) < \infty$. This completes the proof. \square

Lemma 4.3. *Let (M, ω) be a closed symplectic manifold. Let $(U, \mathcal{H}) \in \mathfrak{T}(M, \omega, g)$, and $\xi_1, \xi_2 : [0, 1] \rightarrow [0, 1]$ be two smooth functions. Assume that ξ_1 is monotone. Then there exists a constant C which depends on (U, \mathcal{H}) such that,*

$$D^1((U, \mathcal{H})^{\xi_1}, (U, \mathcal{H})^{\xi_2}) \leq C \|\xi_1 - \xi_2\|_{ham}.$$

We shall give a complete proof of this lemma later on. The following result is an immediate consequence of Lemma 4.3.

Lemma 4.4. *Let (M, ω) be a closed symplectic manifold. Let (U^i, \mathcal{H}^i) be a Cauchy sequence in D^1 , and $\xi_1, \xi_2 : [0, 1] \rightarrow [0, 1]$ be two monotone smooth functions. Given $\epsilon > 0$, there exists two positive constants $\delta = \delta(\{(U^i, \mathcal{H}^i)\})$, and $j_0 = j_0(\{(U^i, \mathcal{H}^i)\})$, such that : if ξ_1, ξ_2 satisfy $\|\xi_1 - \xi_2\|_{ham} < \delta$, then*

$$D^1((U^i, \mathcal{H}^i)^{\xi_1}, (U^i, \mathcal{H}^i)^{\xi_2}) < \epsilon,$$

for all $i \geq j_0$.

Proof. Since the sequence (U^i, \mathcal{H}^i) is Cauchy in D^1 , one can find an integer j_0 large enough such that $D^1((U^i, \mathcal{H}^i)^{\xi_1}, (U^{j_0}, \mathcal{H}^{j_0})^{\xi_1}) < \epsilon/3$ for all $i \geq j_0$. Next, we choose $\delta = \epsilon/3C$ where C is obtained by applying Lemma 4.3 to $(U^{j_0}, \mathcal{H}^{j_0})$, and derive that

$$\begin{aligned} D^1((U^i, \mathcal{H}^i)^{\xi_1}, (U^i, \mathcal{H}^i)^{\xi_2}) &\leq D^1((U^i, \mathcal{H}^i)^{\xi_1}, (U^{j_0}, \mathcal{H}^{j_0})^{\xi_1}) \\ &+ D^1((U^{j_0}, \mathcal{H}^{j_0})^{\xi_1}, (U^{j_0}, \mathcal{H}^{j_0})^{\xi_2}) + D^1((U^{j_0}, \mathcal{H}^{j_0})^{\xi_2}, (U^i, \mathcal{H}^i)^{\xi_2}) \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3, \end{aligned}$$

as long as $\|\xi_1 - \xi_2\|_{ham} < \delta$, and $i \geq j_0$. This completes the proof. \square

The following result is the symplectic analogue of a slight variation of the $L^{(1, \infty)}$ -approximation lemma found in [10]. It shows that any symplectic isotopy can be approximated arbitrarily closely in D^1 by a boundary flat symplectic path (with the same extremities) so that they are also close in \bar{d} .

Lemma 4.5. *Let (M, ω) be a closed symplectic manifold. Let $\Phi = \phi_{(U, \mathcal{H})}$ be a symplectic isotopy, and let ϵ be a positive real number. Then, there exists a boundary flat symplectic isotopy $\Psi = \psi_{(V, \mathcal{K})}$ with the same extremities than Φ such that $D^1((U, \mathcal{H}), (V, \mathcal{K})) < \epsilon$, and $\bar{d}(\Psi, \Phi) < \epsilon$.*

Proof. Let ϵ be a positive real number. We consider $\xi : [0, 1] \rightarrow [0, 1]$ to be any smooth positive and increasing function, which is constant on the intervals $[0, \delta]$ and $[1 - \delta, 1]$ where $0 < \delta < 1/13$. Next, we define (V, \mathcal{K}) to be the element $(U, \mathcal{H})^\xi$ as explained in Section 2.5. It follows from the definition of the curve ξ that the symplectic isotopy $\psi_{(V, \mathcal{K})}$ is boundary flat and has the same extremities than $\phi_{(U, \mathcal{H})}$. Applying Lemma 4.3 with $\xi_1 = id$ and $\xi_2 = \xi$, we deduce that $D^1((U, \mathcal{H}), (V, \mathcal{K})) \leq C \|\xi - id\|_{ham}$ where C is the constant in Lemma 4.3 which depends only on (U, \mathcal{H}) . On the other hand, since the maps $(t, x) \mapsto \phi_{(U, \mathcal{H})}^t(x)$ and $(t, x) \mapsto \phi_{(U, \mathcal{H})}^{-t}(x)$ are Lipschitz continuous, it turns out that there exists a constant $l_0 > 0$ which depends only on (U, \mathcal{H}) such that $\bar{d}(\phi_{(U, \mathcal{H})}, \psi_{(V, \mathcal{K})}) \leq l_0 \|\xi - id\|_{C^0} < l_0 \|\xi - id\|_{ham}$. To complete the proof, it suffices to choose the curve ξ so that $\|\xi - id\|_{ham} \leq \min\{\epsilon/C; \epsilon/l_0; \epsilon\}$. This completes the proof. \square

Proof of Lemma 4.3. In the following Φ represents the symplectic isotopy generated by (U, \mathcal{H}) .

- Step (1). Consider the normalized function $V = U^{\xi_1} - U^{\xi_2}$, and compute

$$|V_t| = |\dot{\xi}_1(t)U_{\xi_1(t)} - \dot{\xi}_2(t)U_{\xi_2(t)}| \leq |\dot{\xi}_1(t)||U_{\xi_1(t)} - U_{\xi_2(t)}| + |U_{\xi_2(t)}||\dot{\xi}_1(t) - \dot{\xi}_2(t)|,$$

for each t . Since M is compact, we use the Lipschitz nature of the smooth map $(t, x) \mapsto U_t(x)$ to derive the existence of a constant $k_0 > 0$ depending on U such that $\max_{x \in M} |U_t(x) - U_s(x)| \leq k_0 |t - s|$ for all $t, s \in [0, 1]$. This yields

$$0 \leq \max_{x \in M} V_t(x) \leq k_0 |\dot{\xi}_1(t)| |\xi_1(t) - \xi_2(t)| + \max_x (U_t(x)) |\dot{\xi}_1(t) - \dot{\xi}_2(t)|.$$

Similarly, we derive that

$$0 \leq -\min_{x \in M} V_t(x) \leq k_0 |\dot{\xi}_1(t)| |\xi_1(t) - \xi_2(t)| - \min_x (U_t(x)) |\dot{\xi}_1(t) - \dot{\xi}_2(t)|.$$

It follows straight from the above estimates that

$$(4.4) \quad \int_0^1 \text{osc}(V_t) dt \leq 2k_0 \max_t |\xi_1(t) - \xi_2(t)| + \max_t (\text{osc}(U_t)) \int_0^1 |\dot{\xi}_1(t) - \dot{\xi}_2(t)| dt.$$

- Step (2). We set $\mathcal{K} = \mathcal{H}^{\xi_1} - \mathcal{H}^{\xi_2}$, and compute

$$|\mathcal{K}_t| = |\dot{\xi}_1(t)\mathcal{K}_{\xi_1(t)} - \dot{\xi}_2(t)\mathcal{K}_{\xi_2(t)}| \leq |\xi_1(t) - \xi_2(t)||\dot{\xi}_1(t)| + |\mathcal{H}_{\xi_2(t)}||\dot{\xi}_1(t) - \dot{\xi}_2(t)|,$$

for each t . The Lipschitz nature of the smooth map $t \mapsto \mathcal{H}_t$, tells us that there exists a constant $c_0 > 0$ which depends on \mathcal{H} such that $|\mathcal{H}_t - \mathcal{H}_s| \leq c_0 |t - s|$ for all $s, t \in [0, 1]$. This yields

$$(4.5) \quad |\mathcal{K}_t| \leq c_0 |\xi_1(t) - \xi_2(t)| |\dot{\xi}_1(t)| + |\mathcal{H}_{\xi_2(t)}| |\dot{\xi}_1(t) - \dot{\xi}_2(t)|.$$

Integrating (4.5) in the variable t yields,

$$(4.6) \quad \int_0^1 |\mathcal{K}_t| dt \leq 2c_0 \max_t |\xi_1(t) - \xi_2(t)| + \max_t |\mathcal{H}_t| \int_0^1 |\dot{\xi}_1(t) - \dot{\xi}_2(t)| dt$$

Adding (4.4) and (4.6) together we get

$$(4.7) \quad D_0((U, \mathcal{H})^{\xi_1}, (U, \mathcal{H})^{\xi_2}) \leq 4 \max\{k_0 + c_0, \max_t (|\mathcal{H}_t| + \text{osc}(U_t))\} \|\xi_1 - \xi_2\|_{ham}$$

- Step (3). On the other hand, we compute

$$\overline{(U, \mathcal{H})^{\xi_j}} = (-U^{\xi_j} \circ \Phi^{\xi_j} - \tilde{\Delta}(\mathcal{H}^{\xi_j}, \Phi^{\xi_j}), -\mathcal{H}^{\xi_j}),$$

for each $j = 1, 2$ and derive from (4.2) that

$$\begin{aligned} D_0(\overline{(U, \mathcal{H})^{\xi_1}}, \overline{(U, \mathcal{H})^{\xi_2}}) &\leq \int_0^1 \text{osc}(\tilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_1}) - \tilde{\Delta}_t(\mathcal{H}^{\xi_2}, \Phi^{\xi_2})) dt \\ &+ \int_0^1 \text{osc}(U_t^{\xi_2} - U_t^{\xi_1}) + |\mathcal{H}_t^{\xi_1} - \mathcal{H}_t^{\xi_2}| dt + \int_0^1 \text{osc}(U_t^{\xi_1} \circ \Phi^{\xi_2}(t) - U_t^{\xi_1} \circ \Phi^{\xi_1}(t)) dt, \\ &\leq \int_0^1 \text{osc}(\tilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_1}) - \tilde{\Delta}_t(\mathcal{H}^{\xi_2}, \Phi^{\xi_2})) dt + D_0((U, \mathcal{H})^{\xi_1}, (U, \mathcal{H})^{\xi_2}) \\ &\quad + k_1 \|\xi_1 - \xi_2\|_{ham}. \end{aligned}$$

In the above estimates, to obtain the quantity $k_1 \|\xi_1 - \xi_2\|_{ham}$ we use the Lipschitz nature of the maps $(x, t) \mapsto U_t(x)$, $(x, t) \mapsto \Phi^{-1}(t)(x)$, and $(x, t) \mapsto \Phi(t)(x)$ to derive the existence of a constant $k_1 > 0$ depending on Φ such that $\int_0^1 \text{osc}(\dot{\xi}_1(t)U_{\xi_1(t)} \circ \Phi^{\xi_1}(t) - \dot{\xi}_1(t)U_{\xi_1(t)} \circ \Phi^{\xi_2}(t)) dt \leq k_1 \|\xi_1 - \xi_2\|_{ham}$. We derive from triangle inequality that

$$\begin{aligned} \int_0^1 \text{osc}(\tilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_1}) - \tilde{\Delta}_t(\mathcal{H}^{\xi_2}, \Phi^{\xi_2})) dt &\leq \int_0^1 \text{osc}(\tilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_1}) - \tilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_2})) dt \\ &\quad + \int_0^1 \text{osc}(\tilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_2}) - \tilde{\Delta}_t(\mathcal{H}^{\xi_2}, \Phi^{\xi_2})) dt. \end{aligned}$$

But, applying Lemma 4.2 to \mathcal{H} and Φ^{ξ_2} yields

$$\int_0^1 \text{osc}(\tilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_2}) - \tilde{\Delta}_t(\mathcal{H}^{\xi_2}, \Phi^{\xi_2})) dt \leq B_1 \|\xi_1 - \xi_2\|_{ham},$$

where the positive constant B_1 depends on Φ , while it follows from the proof of Lemma 3.8-(1) that

$$\begin{aligned} &\int_0^1 \text{osc}(\tilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_1}) - \tilde{\Delta}_t(\mathcal{H}^{\xi_1}, \Phi^{\xi_2})) dt \\ &= \int_0^1 \text{osc}\left(\int_{\min\{\xi_1(t), \xi_2(t)\}}^{\max\{\xi_1(t), \xi_2(t)\}} \dot{\xi}_1(t) \mathcal{H}_{\xi_1(t)}(\dot{\psi}_u) \circ \psi_u du\right) dt \\ &\leq 2 \sup_{s, t, x} |\mathcal{H}_t(\dot{\psi}_s)(x)| \|\xi_1 - \xi_2\|_{C^0} \int_0^1 \dot{\xi}_1(t) dt \\ &\leq B_2 \|\xi_1 - \xi_2\|_{ham}, \end{aligned}$$

where $B_2 = 2 \sup_{s, t, x} |\mathcal{H}_t(\dot{\psi}_s)(x)|$ depends on Φ . Hence, we derive from the above statements that

$$\begin{aligned} D_0(\overline{(U, \mathcal{H})^{\xi_1}}, \overline{(U, \mathcal{H})^{\xi_2}}) &\leq D_0((U, \mathcal{H})^{\xi_1}, (U, \mathcal{H})^{\xi_2}) + k_1 \|\xi_1 - \xi_2\|_{ham} \\ &\quad + B_1 \|\xi_1 - \xi_2\|_{ham} + B_2 \|\xi_1 - \xi_2\|_{ham}. \end{aligned}$$

• Step (4). Since

$$D^1((U, \mathcal{H})^{\xi_1}, (U, \mathcal{H})^{\xi_2}) = D_0(\overline{(U, \mathcal{H})^{\xi_1}}, \overline{(U, \mathcal{H})^{\xi_2}}) + D_0((U, \mathcal{H})^{\xi_1}, (U, \mathcal{H})^{\xi_2})/2,$$

we derive from step (2) and step (3) that

$$D^1((U, \mathcal{H})^{\xi_1}, (U, \mathcal{H})^{\xi_2}) \leq C \|\xi_1 - \xi_2\|_{ham},$$

where $C = B_1 + B_2 + k_1 + 4 \max\{k_0 + c_0, \max_t(|\mathcal{H}_t| + \text{osc}(U_t))\} < \infty$. This completes the proof. \square

5. APPLICATIONS

From the topological point of view, Lemma 3.8-(2) suggests that on a closed symplectic manifold any smooth family of harmonic 1-forms \mathcal{H} gives rise to a nontrivial Hamiltonian path which is small in the Hofer norms. As a matter of fact, let ϵ be an arbitrary positive real number. We define a C^0 -neighborhood $\mathcal{W}(\epsilon, \mathcal{H}, r(g), Id)$ of the identity map by

$$\mathcal{W}(\epsilon, \mathcal{H}, r(g), Id) := \{\Psi \in Iso(M, \omega) \mid \bar{d}(\Psi, Id) \leq \min(r(g), \epsilon/[4 \max_t |\mathcal{H}_t| + 1])\}.$$

According to Polterovich (Proposition 5.2.A, [11]) the set $\mathcal{W}(\epsilon, \mathcal{H}, r(g), Id)$ contains at least a nontrivial Hamiltonian loop. On the other hand, Lemma 3.8-(2) suggests that for all $\Psi \in \mathcal{W}(\epsilon, \mathcal{H}, r(g), Id)$ we have $\int_0^1 osc(\tilde{\Delta}_t(\mathcal{H}, \Psi))dt < \epsilon$. But, it follows from the proof of Lemma 3.8-(2) that we also have $\max_t osc(\tilde{\Delta}_t(\mathcal{H}, \Psi)) < \epsilon$. Hence, the Hofer norms of the Hamiltonian path generated by $\tilde{\Delta}(\mathcal{H}, \Psi)$ are bounded from above by ϵ . Furthermore, the mapping $\Delta_{\mathcal{H}} : \Psi \mapsto \tilde{\Delta}(\mathcal{H}, \Psi)$ maps continuously $\mathcal{W}(\epsilon, \mathcal{H}, r(g), Id)$ into a C^0 -neighborhood of the trivial function. This agrees with Lemma 3.2 found in [2].

The following result is a consequence of Theorem 3.12. It compares symplectic paths to mechanical motions. In particular, it tells us how a closed Hamiltonian orbit behaves (or winds) in \mathbb{T}^2 .

Lemma 5.1. *Consider the 2-dimensional revolution torus \mathbb{T}^2 equipped with its standard symplectic form ω . Let α be a non exact closed 1-form over \mathbb{T}^2 . Consider R to be the mechanical motion represented by a complete rotation about the principal axis of \mathbb{T}^2 . Then, either R cannot be represented by the symplectic flow generated by α or there is no meridian circle in \mathbb{T}^2 which is an orbit of a Hamiltonian loop over (\mathbb{T}^2, ω) .*

Proof : Assume that R can be represented by the symplectic flow (θ^t) generated by α and that there exists a Hamiltonian loop $\Phi = (\phi^t)$ whose an orbit is a meridian circle C_0 in \mathbb{T}^2 . Then by assumption there exists a point $z \in M$ such that the path $C_0 : t \mapsto \phi^t(z)$ is a meridian circle in \mathbb{T}^2 . According to Lemma 3.12, we must have, $\Delta_1(\alpha, \Phi)(z) = 0$, since Φ is Hamiltonian. On the other hand,

$$\Delta_1(\alpha, \Phi)(z) = \int_{C_0} \left(\int_0^1 \iota(\dot{\theta}^t) \omega dt \right) = \int_0^1 \int_0^1 \omega(\dot{\theta}^t(C_0(s)), \dot{C}_0(s)) dt ds = \int_{[0,1] \times [0,1]} (\Theta_{C_0})^* \omega,$$

where

$$\Theta_{C_0} : [0, 1] \times [0, 1] \rightarrow M, (t, s) \mapsto \theta^t(\phi^s(z)).$$

Then, we see that $\Delta_1(\alpha, \Phi)(z)$ is the algebraic value of the volume of the set $\{\theta^s(\phi_t(z)) \mid 0 \leq t, s \leq 1\}$, which is nothing than \mathbb{T}^2 . That is,

$$0 = \Delta_1(\alpha, \Phi)(z) = Vol(\mathbb{T}^2) \neq 0.$$

This is a contradiction. The claim follows. \square

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REFERENCES

- [1] A. Banyaga, : *Sur la structure de difféomorphismes qui préservent une forme symplectique*, comment. Math. Helv. **53** (1978) pp 174 — 2227.
- [2] A. Banyaga and S. Tchiuaga : *The group of strong symplectic homeomorphisms in L^∞ -metric*, Adv. Geom. **14** (2014), no. 3, 523539.
- [3] A. Banyaga : *A Hofer-like metric on the group of symplectic diffeomorphisms*, Symplectic topology and measure preserving dynamical systems, (2010), pp. 123.
- [4] A. Banyaga : *On the group of strong symplectic homeomorphisms*, C. R. Math. Acad. Sci. Paris **346** (2008), no. 15 — 16, 867872.
- [5] A. Banyaga, : *Sur la structure de difféomorphismes qui préservent une forme symplectique*, comment. Math. Helv. **53** (1978) pp 174 — 2227.
- [6] M. Hirsch : *Differential Topology*, Graduate Texts in Mathematics, no. 33, Springer Verlag, New York-Heidelberg, **3** (1976) corrected reprint (1994).
- [7] H. Hofer : *On the topological properties of symplectic maps*, Proc. Royal Soc. Edinburgh **115A**(1990), pp 25 — 38.
- [8] H. Hofer and E. Zehnder : *Symplectic invariants and Hamiltonian dynamics*, Birkhauser Advanced Texts, Birkhauser Verlag (1994)..
- [9] F. Lalonde and D. McDuff : *The geometry of symplectic energy*, Ann. of Math. **141**(1995), 711-727.
- [10] Y-G. Oh and S. Müller : *The group of Hamiltonian homeomorphisms and C^0 -symplectic topology*, J. Symp. Geometry **5**(2007) 167 — 225.
- [11] L. Polterovich : *The Geometry of the Group of Symplectic Diffeomorphism*, Lecture in Mathematics ETH Zürich, Birkhäuser Verlag, Basel-Boston (2001).
- [12] S. Tchiuaga : *Some Structures of the Group of Strong Symplectic Homeomorphisms*, Global Journal of Advanced Research on Classical and Modern Geometry. Vol.2, Issue 1, pp.36-49, (2013)
- [13] F. Warner : *Foundation of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York, (1983).

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